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ABSTRACT

This paper establishes existence of subgame perfect equilibrium in pure strategies for a general class of sequential multi-lateral bargaining games, without assuming a stationary setting. The only required hypothesis is that utility functions are continuous on the space of economic outcomes. In particular, no assumption on the space of feasible payoffs is needed. The result covers arbitrary and even time-varying bargaining protocols (acceptance rules), externalities, and other-regarding preferences. As a side result, we clarify the meaning of assumptions on "continuity at infinity."

1. Introduction

Distributional conflicts stand at the core of economics. From budget negotiations among institutional agents to collective or individual wage agreements, from cost-sharing decisions for the financing of public goods to asset liquidations and bankruptcies, such conflicts give rise to rich strategic problems. *Bargaining* models and procedures are rightly viewed as the main tool for their study and resolution. A vast literature has addressed this issue by applying techniques from both cooperative and non-cooperative game theory, with the former providing important axiomatic characterizations of appealing solutions (e.g. Nash, 1950; Kalai and Smorodinsky, 1975; Thomson, 1981) and the latter allowing for explicit procedural analyses taking into account the timing of offers (e.g. Rubinstein, 1982; Shaked and Sutton, 1984; Binmore, 1987).

A particularly important milestone was set by the canonical model of bilateral sequential bargaining (Stahl, 1972; Rubinstein, 1982), which can be viewed as a link between these two approaches, showing that with small enough frictions the equilibrium of the strategic game approximates the cooperative Nash bargaining solution (Nash, 1950). Accordingly, this model has enjoyed widespread popularity and the insights arising from bilateral strategic bargaining have been applied to models of increasing generality, e.g. allowing for more than two bargaining partners. The basic structure of such models specifies a procedure by which a player makes a proposal and the rest of the players collectively decide whether to accept or reject it; in case of rejection another player gets to make a new proposal. Specific models vary in many dimensions, ranging from the order of proposals to the characteristics of individual utilities and the collective acceptance rule, a particularly crucial element whenever there are more than two players.

Even under perfect information, the existence of (subgame perfect) equilibria has always been an issue in this literature. This is because bargaining games are large games. On the one hand, potential proposals are naturally from a continuum, e.g. the division of a resource. On the other hand, in most non-cooperative bargaining models the potential horizon is infinite, so as to not impose artificial, exogenous constraints on the problem (last-period effects). To make progress, the literature has typically concentrated on stationary environments (where each bargaining round is equivalent to previous ones in a well-defined sense, except for the history of offers up to that point) and restricted attention to stationary equilibria. In the recent decades, important equilibrium existence results have been provided by Merlo and Wilson (1995), Banks and Duggan (2000), Kultti and Vartiainen (2010), and Herings and Predtetchinski (2015). All those results, however, focus on stationary environments and restrict to particular subclasses of games by, e.g., making explicit assumptions on the set of feasible payoffs (e.g., convexity) or considering specific collective acceptance rules (typically unanimity).

The assumption of stationarity is presumably appropriate for bargaining in the souq, or for many business deals that are struck in a stationary environment. There are also other bargaining processes, though. Peace negotiations constitute a telling example.

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They get stalled for a while, then breakthroughs are achieved, followed by periods of contemplation, then consensus is achieved swiftly, followed again by stubborn insistence on details, and so on. This is perhaps not surprising, as peace talks typically do not take place in a stationary environment—after all, they tend to happen during an ongoing war. Peace-time negotiations among countries can also display similar phenomena and be affected by changes of government and circumstances. For instance, the "six-party talks" on North Korea's nuclear program went on for six years before they collapsed in 2009, and talks on the possibility of resuming negotiations happened only after a leadership change in North Korea and attempts by several different U.S. governments. Sometimes, the environment appears to be reasonably stationary but the outcome is anything but. Consider, for instance, negotiations on trade liberalization. Those display a stunning amount of non-stationarity—to the extent that they may fail entirely, as witnessed in the Doha round of the WTO that was suspended in 2016 after 14 years of talks (see New York Times, January 1, 2016, p. A22).[1] Negotiation stalemates are not uncommon during mergers and acquisitions, either. Political negotiations among parties, e.g. for government formation, are another example, with many real-life examples including lengthy negotiations and even breakdowns. A recent case is the failure of a three-party coalition to agree on a government agenda in Germany on November 2017, after two months of negotiations (see New York Times, November 20, 2017, p. A9).[2] All these instances point to the need for a bargaining theory that does not rely exclusively on stationarity. Indeed, bargaining theory would be incomplete without a general existence result that allows for non-stationarities and more than two partners, as addressed here and by a small but growing literature (e.g. Li, 2007, 2011; Schweighofer-Kodritsch, 2018).

The objective of this paper is to present a general existence theorem for multilateral sequential bargaining, encompassing a large class of not necessarily stationary problems, and including an algorithm allowing to actually identify subgame-perfect equilibria. A secondary objective is to illustrate the application of the recent general result of Alós-Ferrer and Ritzberger (2016b), which was formulated in abstract topological terms for arbitrary extensive form games. Since the framework extends beyond stationary environments, the analysis does not (and cannot) focus on stationary strategies but rather considers the existence of subgame perfect equilibrium in arbitrary pure strategies. The main result allows for arbitrary acceptance rules, of course including the most widespread rule found in the literature, namely unanimity (e.g. Haller, 1986; Herrero, 1989), but also majority voting (Eraslan and Merlo, 2002) and many other rules considered in applications (Kalandrakis, 2004). Veto rights and dictatorial arrangements are also allowed. Further, the acceptance rules need not be constant over time. The order of proposals also allows for many possibilities. A fixed cyclical order is a popular choice in the received literature, but some natural alternatives are equally plausible. Our result allows for any exogenous order (cyclical or not), but also for endogenous procedures, for example, selecting the first player who rejects the previous offer to become the next proposer (Selten, 1981; Chatterjee et al., 1993; Ray and Vohra, 1999). Finally, differences and asymmetries in utility functions are of course allowed, as they remain important elements affecting equilibrium predictions. Those include but go far beyond differences in discount factors, for the only constraint which will be imposed on utilities is continuity, as we will discuss below. Hence, the bargaining problems studied here need not have any stationary structure other than the fact that the bargaining partners remain the same in all rounds. And even that requirement can be relaxed, since time-dependent acceptance rules allow to declare certain players "dummies" in given periods.

The existence result hence applies to a large class of games. It also guarantees existence of equilibria *in pure strategies*. Further, and in sharp contrast with the literature, it relies on one and only one elementary assumption: continuity of payoffs on the set of feasible economic *outcomes*, i.e. on actual allocations. In particular, and in contrast to previous existence results, no direct assumptions are imposed on the sets of payoffs, and likewise no direct assumptions on the mapping from strategy profiles to bargaining outcomes are made. The result's hypotheses are hence stated on the actual primitives of the model (outcomes and payoff functions). This is important, because continuity of the payoff functions is often straightforward in applications, and it is easier to verify than properties of the space of feasible payoffs. For instance, our result applies directly to other-regarding preferences (Fehr and Schmidt, 1999; Bolton and Ockenfels, 2000), and can be extended to cover utility functions capturing hyperbolic or quasi-hyperbolic discounting (Strotz, 1956; Laibson, 1997).

Moreover, the continuity assumption is an elementary one. Topologically, we take advantage of the fact that the space of instantaneous allocations (vectors of shares) is compact in the Euclidean topology and construct appropriate topologies on the space of economic outcomes based on the Euclidean one. Specifically, what is assumed is that, for each fixed period t, payoff functions are continuous (in the Euclidean sense) with respect to the shares allocated to agents. Additionally, it is assumed that payoffs are "continuous at infinity", in the sense that payoff differences which accrue sufficiently far in the future become negligible in comparison to present payoff differences. This property parallels the assumption in, e.g., Rubinstein (1982, Assumption A4) and amounts to the notion of continuity at infinity introduced by Fudenberg and Levine (1983). In those works, however, continuity at infinity was stated as an additional condition, without an explicitly topological foundation. As a byproduct of our analysis (and a fact of independent interest), however, we prove that continuity at infinity is *not* an additional condition. Rather, we show that payoff functions are continuous with respect to the natural topologies on the space of bargaining outcomes (allocations and times at which they obtain) *if and only if* both conditions mentioned above hold.

It is also worth emphasizing that the existence result, whose proof makes use of the recent abstract result of Alós-Ferrer and Ritzberger (2016b) (see also Alós-Ferrer and Ritzberger, 2016a, 2017a,b), comes with an algorithm which allows to actually identify subgame perfect equilibria for a given model, and which we will illustrate in the examples.

Finally, one should remark that a limitation of the approach presented here is that it is restricted to games without chance moves. The latter can be important for bargaining models, especially if they reflect unstructured bargaining where the order of proposals is unclear. Indeed, the literature (see Section 2) has studied several bargaining protocols including chance moves. Those remain beyond the domain of the present paper. The reason is that general existence results for infinite-horizon games, as the existence result invoked here (Alós-Ferrer and Ritzberger, 2016b), also do not allow for chance moves. Random ordering of proposers and responders, even in perfect information games, may lead to a failure of equilibrium existence. For instance, Britz et al. (2015) show that with a stochastic selection of proposers and a random order of responders under unanimity, stationary

[1] See also Bagwell et al. (2017) for a detailed theoretical and empirical analysis of the Torquay Round (1950–51) of tariff bargaining.

[2] Another example is given by the negotiations after the Spanish general elections of December 2015, which failed to produce a stable coalition, leading to new elections in June 2016. Those were followed by months of negotiations among different parties, which again failed to produce a stable coalition, and ended with the establishment of a minority government in October 2016 (see New York Times, October 30, 2016, p. A12).

subgame perfect equilibria in pure strategies may not exist. This is an instance of the general observation by Luttmer and Mariotti (2003): even with perfect information and a finite horizon equilibrium existence may fail, if chance moves destroy continuity of payoff functions. Hence, the work presented here concentrates on deterministic settings.

The plan of the paper is as follows. Section 2 contains a brief literature overview. Section 3 specifies the class of bargaining models to be studied. Section 4 states the main existence result and provides an intuitive explanation of its proof. Section 5 gives illustrations and discusses a few extensions of the result. Section 6 concludes. The formal proof of the theorem and the necessary topological constructions are relegated to the Appendix.

2. Related literature

2.1. Noncooperative bargaining

It would be an impossible task to review the extensive literature on bargaining models, even if one were to restrict to noncooperative models. This section merely attempts to provide a few pointers to key and recent developments in the area, to better put the main result in perspective.

The starting point of the literature is of course the model of sequential bilateral bargaining (Stahl, 1972; Rubinstein, 1982; Shaked and Sutton, 1984; Binmore, 1987). Models of multilateral ($n \geq 3$) bargaining were quick to follow, starting with Haller (1986) and Herrero (1989). Those authors studied multilateral bargaining games with a unanimity rule and found multiplicity of subgame perfect equilibria if either the bargaining partners are sufficiently patient or voting is simultaneous. At the same time, the link between cooperative and non-cooperative models remained a focal point. For instance, Hart and Mas-Colell (1996) presented a non-cooperative bargaining game applied to a coalitional game that yields the Shapley value and the Nash bargaining solution in special cases. Following on both developments, Krishna and Serrano (1996) recursively extended the bilateral bargaining model to a multilateral problem and established links to cooperative solutions.

Most general existence results for infinite-horizon multilateral bargaining have concentrated on subgame perfect equilibria in stationary strategies and the particular case of unanimity rules. Further, such results usually require additional, explicit assumptions on the set of feasible payoffs. For instance, the seminal works of Merlo and Wilson (1995) and Banks and Duggan (2000) for the unanimity rule establish existence of (pure-strategy) stationary equilibria when the set of feasible payoffs is compact, convex, and comprehensive from below. More recently Kultti and Vartiainen (2010) demonstrate that when the utility possibility set is compact, convex, and strictly comprehensive and the Pareto frontier is differentiable, all stationary subgame perfect equilibrium outcomes converge to the Nash bargaining solution as the delay between proposals vanishes.

Further results along these lines have been recently obtained by Britz et al. (2010, 2014, 2015) and Herings and Predtetchinski (2015, 2016). Britz et al. (2010) study the convergence of stationary subgame perfect equilibrium payoffs as the cost of delay becomes negligible for multilateral sequential bargaining with action-independent proposers. Britz et al. (2014) provide an equilibrium existence result for stationary strategies under unanimity with action-dependent proposers. Herings and Predtetchinski (2015) show existence of stationary equilibria when feasible payoffs form a set that is closed and comprehensive from below and utility functions are bounded, also for the unanimity rule. Herings and Predtetchinski (2016) establish existence and uniqueness of equilibrium for unanimity bargaining in stationary strategies under monotonicity constraints.

Our work concentrates on bargaining games without chance moves. As mentioned in the introduction, Britz et al. (2015) show that stationary subgame perfect equilibria in pure strategies may fail to exist in bargaining games with a unanimity rule where both proposers and the order of responders are determined randomly. Some positive results, however, have been obtained for particular bargaining models with stochastic elements. For instance, existence is preserved in some bargaining models with random ordering of proposers and responders, as in Britz et al. (2010). Eraslan (2002) considers multilateral sequential bargaining when players differ with respect to their probability to become proposer and their discount factors, and characterizes the set of stationary subgame perfect equilibria. Eraslan and Merlo (2002) allow majority voting and that the surplus evolves stochastically; they find multiplicity of stationary subgame perfect equilibrium payoffs, which may not be efficient. Eraslan and McLennan (2013) consider a random-proposer model with acceptance determined by winning coalitions associated with the proposer and use index theory to demonstrate uniqueness of stationary equilibrium payoffs.

Extensions of the basic multilateral bargaining model have allowed for different (but typically fixed) agreement rules, as majority voting (Eraslan and Merlo, 2002). Kalandrakis (2004, 2006) established existence of stationary subgame perfect equilibria for more general agreement rules. Duggan (2017) establishes existence of stationary equilibria for a class of dynamic games that includes many bargaining models. This development has also extended to the relation with the cooperative approach. For instance, Laruelle and Valenciano (2008) characterize an extension of Nash's bargaining solution for voting rules beyond unanimity. In the existence result presented below, all conceivable agreement rules are allowed, and additionally there is no requirement that the agreement rule should remain fixed over time.

Among the many additional extensions of the basic bargaining setting that have been explored in the literature, one should mention multi-issue bargaining. Assuming a unanimity rule, sequential bargaining about several issues has been studied by Inderst (2000), Busch and Horstmann (2002), and In and Serrano (2003, 2004). This extension can also be encompassed in our setting (see Section 5).

2.2. Equilibrium existence in infinite games

Our existence result builds upon the existence theorem by Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4). This is an abstract result establishing existence of subgame perfect equilibria for arbitrarily large perfect information games, provided the space W of plays (outcomes) can be endowed with some compact and separated (Hausdorff) topology such that the payoff functions defined on the space of plays are continuous with respect to that topology. We remark that, when the separation axiom (Hausdorff) is strengthened to perfectly normal, Theorem 1 of Alós-Ferrer and Ritzberger (2016b) becomes a characterization (see Alós-Ferrer and Ritzberger, 2017b). Therefore, in this sense it is as general as any (topological) existence theorem for perfect information games can become.

A previous existence theorem for large perfect information games is due to Harris (1985). The difference between the result of Harris (1985) and the theorem of Alós-Ferrer and Ritzberger (2016b), which we employ here, is that the former constructs a particular (product) topology on W, while the latter allows for an arbitrary topology on W. Specifically, Harris (1985) starts out with individual topologies for the action spaces corresponding to each individual decision in the game. Assuming those to be compact, the product topology is also compact. However, that space is typically much larger than the actual space of plays (seen as chains of choices in the game), and hence Harris (1985) requires the additional assumption that the latter subspace is a closed subset of the product space. In contrast, Alós-Ferrer and Ritzberger (2016b) works directly with the space of plays. In this sense, the existence theorem of Alós-Ferrer and Ritzberger (2016b) encompasses and supersedes the one of Harris (1985). Further, the result of Alós-Ferrer and Ritzberger (2016b) covers many games and topologies not covered by previous existence theorems as Harris (1985) (for a discussion, see Alós-Ferrer and Ritzberger, 2016b, Example 7 and Section 5). For instance, Theorem 1 of Alós-Ferrer and Ritzberger (2016b) allows for topologies not derived from a Tychonoff product construction, a point also illustrated in Alós-Ferrer and Ritzberger (2017a, Example 2) (see also Alós-Ferrer and Ritzberger, 2016a, Examples 1.2, p. 12, 7.3, p. 166, 7.4, p. 168, and 7.17, p. 209).

This difference, however, is less important for the application at hand, because infinite-horizon bargaining games naturally lend themselves to the use of exactly such a product construction. For this reason, the proof of Theorem 1 could also be based on the existence theorem of Harris (1985). The proof's complexity would remain the same, though. For example, one would need to prove directly that the space of plays is a closed subset of the product action space. Relying on Theorem 1 of Alós-Ferrer and Ritzberger (2016b), however, opens the door to generalizations where payoff functions are continuous with respect to arbitrary topologies on plays (see, e.g., Section 5.5). Further, that result provides an explicit algorithm for identifying pure-strategy subgame perfect equilibria of infinite horizon games, which is different from standard arguments relying on truncated games and which will be illustrated in Section 5.1.

3. A general model for multi-lateral bargaining

The class of models studied here encompasses a wide variety of multi-lateral bargaining games, the bilateral case being nested. Their common feature is that offers are made by some proposer, who may be different each round, and then there is a sequential procedure for the decision on whether or not the proposal is implemented. As discussed above, the bargaining literature has often focused on a unanimity rule for the latter. Although this important case is of course covered, the framework presented here allows for many other procedures as well. For example, implementation could be decided by majority voting with simple or qualified majority, a veto mechanism where some or all partners may be able to block the proposal, or simply a dictatorial rule where a designated person has to agree. Furthermore, the decision procedure may change from one round to the next, as may the identity of the proposer. For instance, the first round may require unanimity for the implementation of the proposal, the second a 90-percent majority, the third an 80-percent majority, and so on until at some point the consent of one participant suffices for implementation. Of course, the bar could also move in the other direction, requiring a higher and higher majority as proposals get rejected. Finally, the decision procedure can depend on the result of the previous bargaining round: for instance, next round's proposer might be the first player to reject the previous proposal.

Formally, a *bargaining game* is a quadruple (I, ρ, ψ, u) consisting of a player set $I = \{1, \dots, n\}$, $n \geq 2$, a bargaining protocol $\rho = \left(r^t\right)_{t=1}^{\infty}$, a sequence of aggregation functions $\psi = \left(\psi^t\right)_{t=1}^{\infty}$, and a vector of utility functions $u = (u_i)_{i\in I}$. These objects will now be explained in detail.

Bargaining takes place over potentially infinitely many *rounds* indexed by $t = 1, 2, \dots$ Each round t begins with a *proposal*

$$a^t = (a_1^t, \dots, a_n^t) \in \Delta = \left\{ (a_1, \dots, a_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} a_i = 1,\; a_i \geq 0 \;\forall i \in I \right\}$$

which specifies the intended shares of the current surplus for all players, where w.l.o.g. the surplus is normalized to 1. (In fact, all arguments would go through if the simplex Δ were replaced by a nonempty and compact subset of some Euclidean space.) All participants learn this proposal and then get to express their opinions or cast their votes sequentially. Votes take a 0–1 form, with 1 meaning acceptance and 0 indicating rejection. At every round t the votes cast by players moving from the second to the nth position form a *voting profile* $b^t = (b_2^t, \dots, b_n^t) \in B = \{0, 1\}^{n-1}$ where, for notational convenience, the subscript indicates the order of play in that round and not the player's name.

Whether or not a proposal $a^t \in \Delta$ is actually implemented at (the end of) round t is determined by an *aggregation function* $\psi^t : B \to \{0, 1\}$. That is, given a voting profile b^t at stage t, the proposal a^t is accepted if $\psi^t\left(b^t\right) = 1$ and rejected if $\psi^t\left(b^t\right) = 0$. The only assumption on ψ^t is that $\psi^t(0, \dots, 0) = 0$ and $\psi^t(1, \dots, 1) = 1$, that is, unanimous decisions are implemented.[3] No other assumption is made.

Example 1. The unanimity rule is given by $\psi^t(b) = 0$ for all $b \neq (1, \dots, 1)$. A strict majority rule would be given by $\psi^t(b) = 1$ if and only if $\sum_{i=2}^{n} b_i > (n-1)/2$. A q-majority rule, $q \in [1/(n-1), 1]$, would specify $\psi^t(b) = 1$ if and only if $\sum_{i=2}^{n} b_i \geq q(n-1)$. One could for instance specify that ψ^t is an $f(t)$-majority rule, with $f : \{1, 2, \dots\} \to [1/n, 1]$ a strictly decreasing function of t. In such an example, the acceptance threshold for a decision would be lowered gradually over time, perhaps in an attempt to ensure a timely decision.

If round t's proposal is rejected, the game continues to round $t+1$. Once a proposal is accepted, the game ends. Potentially, the game can run forever (if proposals are always rejected).

The order of votes and the identity of the proposer at round t are determined by a *bargaining protocol* as follows. For each $t = 1, 2, \dots$ and each $j = 1, \dots, n$, let $r_j^t : B^{t-1} \to I$ determine the order of play at t. That is, given the history of previous votes $\bar{b} \in B^{t-1}$, $r_1^t(\bar{b}) \in I$ is the player acting as the proposer at t, hence choosing a^t, while players $r_2^t(\bar{b})$ to $r_n^t(\bar{b})$ are moving second to last, hence casting votes b_2^t to b_n^t, respectively. For instance, $r_1^t(\bar{b}) = i$ means that player $i \in I$ gets to make a proposal, and $r_3^t(\bar{b}) = j$ that player $j \in I$ is the second to cast her vote. The *bargaining protocol* is given by $\rho = \left(r^t\right)_{t=1}^{\infty}$, where $r^t = (r_1^t, \dots, r_n^t)$ is such that $\{r_j^t(\bar{b})\}_{j=1}^{n} = I$ for all $\bar{b} \in B^{t-1}$ and all t. The dependence of r^t on B^{t-1} allows to encompass protocols where the order of play depends on previous voting decisions (but not on the current proposal), as Example 3 illustrates. However, if in a particular example r^t is independent of the order of play and the votes in the previous periods, we will simply write r_j^t rather than $r_j^t(\cdot)$.

Example 2. If $n = 2$, the well-known bilateral bargaining game with alternating proposers (Rubinstein, 1982; Shaked and Sutton, 1984; Binmore, 1987) is obtained by setting $r_1^t = 1$ for all odd t, $r_1^t = 2$ for all even t, and $\psi^t(1) = 1$ and $\psi^t(0) = 0$ for all t. For $n > 2$, a multi-lateral bargaining protocol with alternating proposers can be specified setting $r_1^t = t \mod n$. This could be combined with e.g. a unanimity rule or a simple majority rule as above to obtain standard examples.

Histories of play will just contain the previous offers a^t and previous voting decisions b^t. Note that there is no need to record the actual order of play within a given period t, since the bargaining protocol, which is part of the description of the game, allows to reconstruct that order from the previous voting decisions. The

[3] This assumption is natural, but strictly speaking not needed. It would be enough to assume that ψ^t is not constant.

following example takes advantage of this fact to encompass an action-dependent voting order where the first player to reject the previous offer becomes the next proposer. The example makes transparent why the bargaining protocol needs to depend on all previous rounds, because the name of "the first player to reject" at t depends on the order of votes at t, which in turn depends on the order at $t-1$.

Example 3. Let $r_j^1 = j$ for all $j = 1, \ldots, n$ (which just means that players are named according to their order of play in the initial period). For each $t \geq 1$ and each $(b^1, \ldots, b^t) \in B^t$ with $\psi^t(b^t) = 0$, let

$$r_1^{t+1}(b^1, \ldots, b^t) = \min \left\{ r_j^t(b^1, \ldots, b^{t-1}) \;\middle|\; b_j^t = 0,\ j = 2, \ldots, n \right\}$$

and

$$r_j^{t+1}(b^1, \ldots, b^t) = \begin{cases} j-1 & \text{if } j \leq r_1^{t+1}(b^1, \ldots, b^t) \\ j & \text{if } j > r_1^{t+1}(b^1, \ldots, b^t) \end{cases}$$

for all $j = 2, \ldots, n$. For $(b^1, \ldots, b^t) \in B^t$ with $\psi^t(b^t) = 1$, arbitrarily (and inconsequentially) fix $r^t(b^1, \ldots, b^t) = r^1$. This procedure assigns as proposer the first player to vote against the proposal in the previous period, and lets all other players vote in the fixed order derived from $\{1, \ldots, n\}$.

The bargaining protocol is deterministic. As explained before, a stochastic dependence on previous votes may interfere with continuity of payoffs (Luttmer and Mariotti, 2003), hence our existence result does not apply to that case (see also Britz et al., 2015). A similar comment applies to simultaneous moves.

These specifications define a perfect information game among the n bargaining partners. A *play* in this game is a complete sequence of offers and voting profiles, from the beginning to eventual acceptance, including sequences of infinite length where no offer is ever accepted. The set of plays is given by the union

$$W = \left(\bigcup_{T=1}^{\infty} W^T \right) \bigcup W^\infty$$

where plays that end after a finite number of T rounds with acceptance are

$$W^T = \left\{ \left(a^t, b^t\right)_{t=1}^T \;\middle|\; \psi^T\left(b^T\right) = 1,\ \psi^t\left(b^t\right) = 0\ \forall t < T \right\}$$

and the infinite plays of perpetual disagreement are

$$W^\infty = \left\{ \left(a^t, b^t\right)_{t=1}^\infty \;\middle|\; \psi^t\left(b^t\right) = 0\ \forall t \right\}.$$

The specifications above suffice to define the tree of this extensive form (see, e.g., Alós-Ferrer and Ritzberger, 2016a). In particular, the *nodes* in the tree are those sets of plays that share a fixed initial segment (formal definitions are provided in the Appendix). Due to perfect information the players' *choices* are the (immediate) successor nodes of their decision points. That is, a player active at a node simply chooses among successor nodes, which represent the possible options (proposals or votes).

In general the set W of plays is the appropriate domain for the players' preferences. Yet, in the main part of the paper we take a "non-procedural" stance by assuming that players care only about the ultimate distribution of the surplus and about when agreement was reached—not about how. (See Section 5 for extensions to a "procedural" approach.) In particular, it is assumed that the players' utility functions are defined on the set

$$Z = (\Delta \times \{1, 2, \ldots\}) \cup \{\infty\}$$

of *outcomes*. A pair $(a, t) \in \Delta \times \{1, 2, \ldots\}$ amounts to an agreement on the distribution $a \in \Delta$ at round $t = 1, 2, \ldots$; the outcome ∞ corresponds to perpetual rejections. Accordingly, the players' preferences are represented by *utility functions* (or payoff functions) $u_i : Z \to \mathbb{R}$ for all $i \in I$. The only assumption on those will be continuity. In particular, utility functions need not be monotonic. Further, no assumption on convexity of the set of feasible payoffs is needed (in contrast to Merlo and Wilson, 1995; Banks and Duggan, 2000; Britz et al., 2014).

4. A general existence result

The main hypothesis for the existence theorem in this paper will be *continuity* of the utility functions. But clarifying what this ought to mean is non-trivial, because continuity only makes sense with respect to a topology. Within an infinite-horizon game (and, actually, within any extensive form game), preferences are defined on full histories of choices along the game, that is, on the space of plays W. Hence, utility functions are *a priori* defined on a very large, infinite-dimensional space, and continuity, as required by abstract existence theorems, ultimately refers to a topology on that space.

Within a bargaining game, however, it is natural to restrict attention to the smaller set of outcomes Z, defined above. This poses the problem that continuity of utility functions with respect to Z is disentangled from the relevant topological continuity, that is, continuity of the function assigning payoffs to plays. One key result in our analysis (Proposition B.1 in the Appendix), however, is that for a natural topology on W, continuity of the mapping assigning payoffs to plays can be fully characterized by two properties of the utility functions $u_i : Z \to \mathbb{R}$. That is, in the current setting, continuity of utilities on Z has two parts (as in Rubinstein, 1982, Assumption A4). This is captured by the following definition.

Definition 1. In a bargaining game, the payoff function $u_i : Z \to \mathbb{R}$ of a player i is said to be continuous if

(i) for each $t = 1, 2, \ldots$, the function $u_i^t : \Delta \to \mathbb{R}$ given by $u_i^t(a) = u_i(a, t)$ for each $a \in \Delta$ is continuous (with respect to the Euclidean topologies on Δ and $\mathbb{R}$), and
(ii) for each $\varepsilon > 0$ there exists $T \in \{1, 2, \ldots\}$ such that for all $a \in \Delta$ and all $t \geq T$, $|u_i(a, t) - u_i(\infty)| < \varepsilon$.

Part (i) is simply continuity on Δ (with respect to the topology induced by the Euclidean metric). That is, for all $t = 1, 2, \ldots$ and every $\varepsilon > 0$ there is $\delta > 0$ such that $\|a - a'\| < \delta$ implies $|u_i(a, t) - u_i(a', t)| < \varepsilon$. Part (ii) is "continuity at infinity" (Fudenberg and Levine, 1983), which is called for because of the possibility of perpetual disagreement. Intuitively, the second part says that sufficiently late agreements will make almost no difference. It is worth emphasizing that, because of the characterization result just mentioned, continuity at infinity is *not* an additional assumption here, but rather an integral part of continuity with respect to the appropriate topology on plays. This fact extends beyond the bargaining case and clarifies the role of continuity at infinity for infinite-horizon games.

Remark 1. The present definition of continuity at infinity may appear slightly different from the one proposed by Fudenberg and Levine (1983). They required that for every $\varepsilon > 0$ there is T such that whenever two (pure) strategy profiles agree up to T, then the two strategy profiles yield payoffs within ε from each other. In particular, in their framework there was no outcome of eternal continuation corresponding to $\infty \in Z$. *With* such an outcome ∞ the two definitions in fact agree. For, given $\varepsilon > 0$ consider a strategy profile that rejects all proposals up to the T associated with ε by the Fudenberg–Levine criterion. Then there is a second strategy profile that is identical to the first up to T but thereafter always rejects, hence, yields the outcome ∞. If $(a, t) \in Z$ denotes the outcome induced by the first strategy profile, then $|u_i(a,t) - u_i(\infty)| < \varepsilon$, as required. Conversely, given $\varepsilon/2 > 0$ consider two strategy profiles that reject all proposals up to the T associated with $\varepsilon/2$ by the present criterion. Then the payoffs to the two strategy profiles must both be within $\varepsilon/2$ from $u_i(\infty)$. By the triangle inequality it follows that the payoffs to the two strategy profiles must be within ε from each other. Hence, the two definitions coincide.

Impatience, as implied e.g. by geometric discounting and bounded utility functions, is a sufficient condition for continuity at infinity. The reason is that discounting will drive all payoffs to zero as time goes on, which must then also be the utility from perpetual disagreement. But impatience is a strictly stronger condition than continuity at infinity. The following example illustrates this point, that without continuity at infinity subgame perfect equilibria may not exist—and that the present set-up covers highly non-stationary cases.

Example 4. Rumor has it that some committees are kept busy negotiating for the sole purpose of avoiding to be dissolved, even though nobody wishes to negotiate forever. This can be expressed by specifying $u_i(a,t) = 1 - \delta^{t-1}$ for some $\delta \in (0,1)$ and all $t = 1, 2, \ldots,$ and $u_i(\infty) = 0$ for all $i \in I$. These utility functions fail continuity at infinity. Let $n = 2$, alternating offers as given by $r_1^t = t \mod 2$, and $\psi^t(0) = 0$ and $\psi^t(1) = 1$. This game has no subgame perfect equilibrium (defined below). For, suppose that there is an equilibrium that ends with agreement in round t. Then the responder r_2^t can do better by rejecting now and accepting two rounds later. If there were an equilibrium with perpetual disagreement, then at every finite t the responder r_2^t could do better by accepting immediately. If instead $u_i(\infty)$ were set to 1 for all $i \in I$, restoring continuity at infinity and existence of equilibrium, the committee would negotiate forever.

A *subgame perfect equilibrium* is a Nash equilibrium that induces a Nash equilibrium in every subgame. Since bargaining games have perfect information, a new subgame begins at every move (non-terminal node) of the tree. The following is the main result of the present paper.

Theorem 1. *Every bargaining game with continuous payoff functions has a subgame perfect equilibrium.*

Due to the generality of the class of games studied here the details of the proof of this theorem are somewhat involved and therefore relegated to the Appendix. Still, the next section provides an intuitive tour of the main ideas. (Readers who are not interested in the construction may skip it.)

4.1. Structure of the proof

The proof consists of four steps. The first, and most laborious, is to endow the set Z of outcomes and the set W of plays with compact separated (Hausdorff) topologies in such a way that the mapping from plays to outcomes is a continuous function $\varphi : W \to Z$. The purpose is to turn the utility functions $u_i : Z \to \mathbb{R}$ that are defined on Z into continuous functions $u_i \circ \varphi : W \to \mathbb{R}$, now defined on plays.

Based on first principles the space of outcomes is the union $Z = (\Delta \times \{1, 2, \ldots\}) \cup \{\infty\}$. However, topologies derived from unions are notorious for being badly behaved, and hence we consider an alternative approach. The topology on Z is derived from considering the auxiliary set $\Delta \times \{1, 2, \ldots, \infty\}$. The simplex Δ carries the natural relative Euclidean topology. The extended natural numbers $\{1, 2, \ldots, \infty\}$ are equipped with the standard one-point compactification topology. This is the topology for which the open sets are those whose complements are either finite or contain the point ∞ (equivalently, a subset is closed if it is either finite or contains ∞). The topology on the product $\Delta \times \{1, 2, \ldots, \infty\}$ is then the product topology (the coarsest topology that makes both projections continuous). To obtain a topology on Z, all elements of $\Delta \times \{1, 2, \ldots, \infty\}$ with second coordinate ∞ are identified to a single equivalence class. The resulting quotient set $\tilde{Z}$ is endowed with the quotient topology, the coarsest topology that makes the projection π continuous (where π is defined by $\pi((a,t)) = (a,t)$ if $t \neq \infty$ and $\pi((a,t)) = [\infty]$ otherwise). The end result corresponds to a one-point compactification of the space $\Delta \times \{1, 2, \ldots\}$. Since $\tilde{Z}$ and the original set Z of outcomes can be identified, this induces a topology on Z. And this topology can be shown to be compact and separated. (Compactness of $\tilde{Z}$, hence Z, is directly inherited from compactness of $\Delta \times \{1, 2, \ldots, \infty\}$; to show that it is separated takes some work.)

To endow the set W of plays with a topology, this is first embedded into the larger space

$$\Omega^\infty = [(\Delta \cup \{*\}) \times (B \cup \{**\})]^\infty$$

of infinite sequences, where $*$ and $**$ denote dummy options, representing that the game has already ended. Again, Δ carries the relative Euclidean topology, which remains compact and separated if the single, discretely separated point $*$ is appended. The finite set $B \cup \{**\}$ is endowed with the discrete topology and is trivially compact and separated. By Tychonoff's theorem the infinite product Ω^∞ is also compact (and Hausdorff is easily established). Yet, this set is too big. Plays correspond only to infinite sequences in the subset $\bar{W} \subseteq \Omega^\infty$ where once an offer is accepted, play actually ends. The set $\bar{W}$ is characterized by the following three conditions.

(S.1) $\tilde{a}^1 \in \Delta$,
(S.2) $\tilde{a}^t = * \Leftrightarrow \tilde{b}^t = **$, for all $t = 1, 2, \ldots,$
(S.3) $\left(\tilde{a}^t, \tilde{b}^t\right) \in \Delta \times B \Leftrightarrow \psi^s\left(\tilde{b}^s\right) = 0 \, \forall s = 1, \ldots, t-1$, for all $t = 1, 2, \ldots$

That is, (S.1), they begin with offers; (S.2), dummy options occur always in both coordinates if at all; and, (S.3), a new bargaining round starts if and only if all previous proposals have been rejected. Call sequences in $\bar{W}$ *bargaining sequences.*

The relevant set, hence, is the set $\bar{W}$ of bargaining sequences. The proof establishes that this set is closed in the set Ω^∞ of all sequences, which in turn implies that it is compact in the relative topology. This topology then defines the appropriate topology, because the set of plays can be fully identified with $\bar{W}$ as follows. Consider the map $\Lambda : W \to \bar{W}$ defined by

$$\Lambda(w) = \left((a^1, b^1), \ldots, (a^T, b^T), (*, **), \ldots, (*, **), \ldots\right)$$

if $w = \left(a^t, b^t\right)_{t=1}^T \in W^T$, $T \neq \infty$, and $\Lambda(w) = w$ if $w \in W^\infty$. Showing that Λ is bijective establishes that the set $\bar{W}$ of bargaining sequences and the set W of plays are isomorphic. Hence, the relative topology on $\bar{W}$ can be used as the compact and separated topology on W.

Once W and Z are endowed with compact separated topologies as described, one can turn to the map assigning outcomes to plays, $\varphi : W \to Z$, which is defined by $\varphi(w) = \left(a^T, T\right)$ if $w = \left(a^t, b^t\right)_{t=1}^T \in W^T$, $T \neq \infty$, and $\varphi(w) = \infty$ otherwise. The first step of the proof is completed by showing that φ is continuous, resulting in continuous payoff functions $u_i \circ \varphi$ on a compact separated space of plays.

With this preparation in place, it is now possible to invoke the general existence theorem by Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4). This result guarantees existence of subgame perfect equilibria for arbitrarily large perfect information games, provided the space of plays is compact and separated and the payoff functions defined on the space of plays are continuous with respect to the topology on plays.

The existence theorem requires three additional hypotheses, and hence the three remaining steps amount to checking them. The second step establishes that the perfect information game is *well-behaved*. This means showing that the set of non-terminal nodes at a given "distance" from the root is partitioned into finitely many cells, each of which consists of decision points of a single player, whose unions are closed in the topology on plays. This is essentially straightforward in the current setting, because at the voting stages only finitely many players move. The only point that requires some work occurs when a proposal is made, because there are terminal nodes at the same distance from the root as the proposers' nodes.

The third step verifies that all nodes of the tree are closed as sets of plays, which intuitively is necessary for the players' optimization problems to be well-defined. The fourth step, finally, establishes that the assignment of immediate predecessors of nodes in the tree constitutes an open map, i.e. takes open sets to open sets. This is equivalent to the assignment of immediate successors being lower hemi-continuous, and essentially allows to "paste" the solutions of individual optimization problems together.

Once these four preparations are in place, all hypotheses of the existence theorem by Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4) are verified and the existence of a subgame perfect equilibrium follows.

5. Extensions and illustrations

This section illustrates the algorithm which underlies the proof of Theorem 1 and offers a few possible extensions of the main result.

5.1. The algorithm

The proof of Theorem 1 invokes the abstract existence theorem by Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4). The proof of the latter is based on an algorithm that iterates the players' expectations about what later players will do until, in the limit, expectations are correct and behavior is optimal with respect to these expectations.

The algorithm works as follows. Players start naïvely, that is, when they decide, they pick a play as if they had full control of all other players moving afterwards. This is the first step. In the second step players develop some anticipation and now reoptimize under the constraints generated by what later players have done in the first step. Hence, they become "smarter" and foresee the choices (from the first step) of later players. This is repeated in the third step. Players now reoptimize under the constraints generated by the choices of later players from the second step, and so on. This iteration has a limit, which is a set, though. The existence theorem mentioned above shows that from this set strategy profiles can be selected which in turn form subgame perfect equilibria. If there is a unique equilibrium, the limit set is a singleton and the algorithm delivers the equilibrium directly. This can be nicely illustrated by bargaining games with a unique equilibrium, e.g., by the classical bilateral alternating-proposer model of Rubinstein (1982).

Example 5. More concretely, let $n = 2$, $r_1^t = t \bmod 2$, $r_2^t = 3 - r_1^t$, and $\psi^t(0) = 0$, $\psi^t(1) = 1$, for all $t = 1, 2, \ldots$, that is, players take turns in making offers and the game ends once the responder has accepted. For simplicity let payoffs be given by $u_i(a, T) = \delta^{T-1} a_i$ for some common discount factor $\delta \in (0, 1)$, for all $a \in \Delta$ and all $T = 1, 2, \ldots$, and by $u_i(\infty) = 0$, for all $i \in I$. Discounting is from one bargaining round to the next.

Obviously, Theorem 1 covers this basic example, which we use now to illustrate the algorithm. The first step works as follows. All proposers ask everything for themselves, $c_1 = 1$, on the assumption that they can force the responders to accept. All responders who are offered more than δ will accept, because they earn more than by rejecting and asking everything for themselves next round. All responders who are offered less than δ will reject, on account of making an accepted counteroffer next period that allocates the whole surplus to them. Therefore, the "critical offer" (from the proposer's viewpoint) for the second step is $c_2 = 1 - \delta$. This is *critical* because it is the only one at which the responder may choose both Yes (1) and No (0). Since all offers give 1 to the proposer and 0 to the responder, by $\delta < 1$ all offers effectively lead to perpetual rejections under first-step behavior and to payoffs $u_i(\infty) = 0$ for $i = 1, 2$.

Now turn to the second step. Since under first-step behavior all offers that allocate less than δ to the responder are rejected forever, in the second step all proposers offer δ to the responder and demand $c_2 = 1 - \delta$ for themselves. Hence, a responder, in a subgame after a proposal that allocates less than δ to her, cannot count on rejecting and asking 1 for herself next round, but must take into account that she can at best get $1 - \delta$ as next round's proposer, which is now worth $\delta(1 - \delta)$ to her. Therefore, she will now accept any offer that leaves the proposer with no more than $1 - \delta(1 - \delta) = 1 - \delta + \delta^2$. The critical offer for the third step is consequently $c_3 = 1 - \delta + \delta^2$. At this proposal the responder is indifferent between accepting and rejecting.

More generally, denote the critical offer from the τth step for the $(\tau + 1)$th step by c_τ. That is, any offer that allocates more than c_τ to the proposer (less than $1 - c_\tau$ to the responder) leads to perpetual rejections, and any offer that allocates less than c_τ to the proposer (more than $1 - c_\tau$ to the responder) is accepted under τth step behavior; only at c_τ both acceptance and rejection are possible. Then in the $(\tau + 1)$th step all proposals will allocate c_τ to the proposer and $1 - c_\tau$ to the responder. But a responder who is confronted with an offer that gives her less than $1 - c_\tau$ anticipates that by rejecting she will only be able to ask c_τ for herself next round, which is now worth δc_τ to her. (If the offer still gives her more than δc_τ, she will now accept.) Therefore, the critical offer at the $(\tau + 1)$th step is $c_{\tau+1} = 1 - \delta c_\tau$. This is a difference equation with initial condition $c_1 = 1$ and the unique solution

$$c_{\tau+1} = \sum_{j=0}^{\tau} (-1)^j \delta^j.$$

It follows that under the algorithm the critical offers converge to

$$c_\infty = \sum_{j=0}^{\infty} (-1)^j \delta^j = \frac{1}{1+\delta},$$

and this must be the offer made by all proposers in any subgame perfect equilibrium; in particular, such an equilibrium is necessarily unique.

5.2. Hyperbolic discounting and multiple selves

Simple textbook examples of bargaining games often use geometric discounting (and $u_i(\infty) = 0$) to ensure continuity at infinity. Theorem 1 makes no such assumptions. In fact, it is even consistent with time-inconsistent choices as those arising from "hyperbolic" discounting (Strotz, 1956) where decision makers today discount the step from today to tomorrow more than the step from tomorrow to the day after tomorrow.[4] Consider, for instance, the case of quasi-hyperbolic discounting (Laibson, 1997), which captures this effect through the sequence of discount factors $1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots$ for $\beta, \delta \in (0, 1)$. The only subtlety that arises for such "time-inconsistent" preferences is that players have to be split into agents—a different agent for each bargaining round. This is because players have different preferences in each consecutive bargaining round. More generally, dynamically inconsistent preferences are often analyzed by splitting an agent into a sequence of temporal selves making choices in a dynamic game; this has given rise to the literature on "multiple selves" (see, e.g., O'Donoghue and Rabin, 1999, 2001; Fudenberg and Levine, 2006, 2012). Technically, this implies that the resulting bargaining game has infinitely many players, and hence it is not covered by Theorem 1. An extension to this case, however, poses no difficulty, because the underlying existence theorem (Alós-Ferrer and Ritzberger, 2016b) invoked in the proof of Theorem 1 actually allows for infinitely many players (that is, agents or selves), provided only finitely many of them move at the same stage (round) of the game. Therefore, any kind of time preference can be accommodated, as long as continuity is preserved.

Example 6. Let $n = 3$, $r_j^t = (t + j - 1) \bmod 3$ for $j = 1, 2, 3$, and $\psi^t(b^t) = 1$ if and only if $b_2^t + b_3^t = 2$ for all $b^t \in \{0, 1\}^2$, for all $t = 1, 2, \ldots$ That is, three players take turns in making proposals and acceptance is by unanimity. Utility functions are given by $u_i(a, 1) = a_i$ and $u_i(a, T) = \beta\delta^{T-1}a_i$ for some $\beta, \delta \in (0, 1)$ for $T = 2, 3, \ldots$, for all $a \in \Delta$ and all $i = 1, 2, 3$, i.e., the next round is discounted by $\beta\delta$ while later rounds $t > 1$ are discounted by $\beta\delta^{t-1}$. Theorem 1 (extended as explained above) applies, and hence the existence of subgame perfect equilibrium is guaranteed. In this case, the equilibrium can be identified through standard arguments.

Let $\bar{v}_i^t$ resp. $\underline{v}_i^t$ denote the supremum resp. the infimum of player i's equilibrium payoffs in any subgame starting in the tth round, for $t = 1, 2, \ldots$ and $i = 1, 2, 3$. All subgames starting in the fourth round are identical to the game starting in the first round. Thus, as i will accept (resp. reject) any offer that gives her more (resp. less) than $\bar{v}_i^t$ (resp. $\underline{v}_i^t$), the usual optimality argument (see Shaked and Sutton, 1984) yields, for $t = 3$ and player 3,

$$\bar{v}_3^3 = 1 - \beta\delta\left(\underline{v}_1^4 + \underline{v}_2^4\right) \text{ and } \underline{v}_3^3 = 1 - \beta\delta\left(\bar{v}_1^4 + \bar{v}_2^4\right),$$

and $\bar{v}_i^3 = \beta\delta\bar{v}_i^4$ and $\underline{v}_i^3 = \beta\delta\underline{v}_i^4$ for players $i = 1, 2$. Further, for $t = 2$,

$$\bar{v}_3^2 = \beta\delta - \beta^2\delta^2\left(\underline{v}_1^4 + \underline{v}_2^4\right) \text{ and } \underline{v}_3^2 = \beta\delta - \beta^2\delta^2\left(\bar{v}_1^4 + \bar{v}_2^4\right)$$

for player 3, while for player 2

$$\bar{v}_2^2 = 1 - \beta\delta\underline{v}_1^3 - \beta\delta\underline{v}_3^3 = 1 - \beta\delta + \beta^2\delta^2\left(\bar{v}_1^4 - \underline{v}_1^4\right) + \beta^2\delta^2\bar{v}_2^4 \text{ and}$$
$$\underline{v}_2^2 = 1 - \beta\delta\left(\bar{v}_1^3 + \bar{v}_3^3\right) = 1 - \beta\delta - \beta\delta\left(\bar{v}_1^4 - \underline{v}_1^4\right) + \beta^2\delta^2\underline{v}_2^4,$$

and $\bar{v}_1^2 = \beta^2\delta^2\bar{v}_1^4$ and $\underline{v}_1^2 = \beta^2\delta^2\underline{v}_1^4$ for player 1. Finally, for $t = 1$,

$$\bar{v}_3^1 = \beta^2\delta^2 - \beta^3\delta^3\left(\underline{v}_1^4 + \underline{v}_2^4\right) \text{ and } \underline{v}_3^1 = \beta^2\delta^2 - \beta^3\delta^3\left(\bar{v}_1^4 + \bar{v}_2^4\right)$$

for player 3, while for player 2

$$\bar{v}_2^1 = \beta\delta - \beta^2\delta^2 + \beta^3\delta^3\left(\bar{v}_1^4 - \underline{v}_1^4\right) + \beta^3\delta^3\bar{v}_2^4 \text{ and}$$
$$\underline{v}_2^1 = \beta\delta - \beta^2\delta^2 - \beta^3\delta^3\left(\bar{v}_1^4 - \underline{v}_1^4\right) + \beta^3\delta^3\underline{v}_2^4,$$

and for player 1

$$\bar{v}_1^1 = 1 - \beta\delta + \beta^3\delta^3\bar{v}_1^4 + \beta^3\delta^3\left(\bar{v}_1^4 - \underline{v}_1^4 + \bar{v}_2^4 - \underline{v}_2^4\right) \text{ and}$$
$$\underline{v}_1^1 = 1 - \beta\delta + \beta^3\delta^3\underline{v}_1^4 + \beta^3\delta^3\left(\bar{v}_1^4 - \underline{v}_1^4\right) - \beta^3\delta^3\left(\bar{v}_2^4 - \underline{v}_2^4\right).$$

Since all subgames beginning in the fourth round are identical to the game that starts in the first round, $\bar{v}_i^1 = \bar{v}_i^4$ and $\underline{v}_i^1 = \underline{v}_i^4$ for all $i = 1, 2, 3$. Solving this linear equation system yields

$$\bar{v}_i^1 = \underline{v}_i^1 = \frac{\beta^{i-1}\delta^{i-1}}{1 + \beta\delta + \beta^2\delta^2}$$

for all $i = 1, 2, 3$. Due to hyperbolic discounting player 3's agent in the first round accepts the offer $a_3^1 = \beta^2\delta^2/(1 + \beta\delta + \beta^2\delta^2)$ even though she would prefer to wait for the third round and ask $1/(1 + \beta\delta + \beta^2\delta^2)$ for herself, which she now values at $\beta\delta^2/(1+\beta\delta+\beta^2\delta^2) > a_3^1$. But she has no control over her agent in the second round who accepts $a_2^2 = \beta\delta/(1+\beta\delta+\beta^2\delta^2)$, which in the second round is as good to him as asking $1/(1+\beta\delta+\beta^2\delta^2)$ in the third round. Therefore, player 3's agent in the first round has to accept an offer that is worth less to her than the discounted value of her own offer in the third round.

A special case of this example is $\beta = 1$, the standard case of geometric discounting. For that case the example shows that with a unanimity rule three-player bargaining yields a unique subgame perfect equilibrium.[5]

5.3. Other-regarding preferences

The specification of utility functions for the general bargaining games studied in this paper only requires continuity of payoffs on the space of bargaining outcomes. There is no requirement whatsoever that payoffs depend only on the own coordinate of the allocation. In particular, players may not only care about their share, as was explicitly assumed e.g. in Rubinstein (1982, Assumption A1). This leaves room for explicitly incorporating externalities and other-regarding preferences, as in the models of Fehr and Schmidt (1999) or Bolton and Ockenfels (2000). That is, as a consequence of Theorem 1, equilibrium existence is guaranteed in any bargaining model where some or all players display other-regarding preferences.

More complex examples are also possible where the preferences of players not only depend on the shares received by other players, but also do so through the actual utility that players derive from the shares. This is illustrated in the following example with three bargaining partners.

Example 7. Let $n = 3$, $r_j^t = (t+j-1) \bmod 3$ for all $j = 1, 2, 3$, and $\psi^t(b^t) = 1$ if and only if $\sum_{j=2}^3 b_j^t \geq 1$ for all $b^t \in \{0, 1\}^2$, for all $t = 1, 2, \ldots$ That is, players take turns in proposing and voting and the acceptance decision is by simple majority voting (since the proposal counts as a Yes-vote). Picture the three players in a triangle, with player 2 to the right of player 1 and player 3 to her left. The payoff functions display other-regarding preferences as follows.

$$u_i(a, T) = \delta^{T-1}a_i + \alpha u_{(i+1) \bmod 3}(a, T) - \alpha^2 u_{(i+2) \bmod 3}(a, T)$$

for some $\alpha, \delta \in (0, 1)$ for all $a \in \Delta$, all $T = 1, 2, \ldots$, and $i = 1, 2, 3$. That is, what matters to a player is not only her share of the surplus, but also how her neighbor to the right feels about the allocation (with weight α), even though she despises a bit the opinion of her neighbor to the left (with weight $-\alpha^2$). Solving the equation system for utility functions yields

$$u_i(a, T) = \frac{\delta^{T-1}\left(a_i + \alpha a_{(i+1) \bmod 3}\right)}{1 + \alpha^3}$$

[4] Schweighofer-Kodritsch (2018) characterizes subgame perfect equilibria in the two-player, alternating offers bargaining game of Rubinstein (1982) when players have time-inconsistent preferences, and shows that non-stationary equilibria arise under certain violations of present bias.

[5] We do not know whether or not this logic extends to the case $n > 3$.

for all $a \in \Delta$, all $T = 1, 2, \dots,$ and $i = 1, 2, 3$, i.e., players care about their own share and their neighbor's. Hence, utilities are continuous and Theorem 1 applies. An equilibrium – one of many, of course – is easily found: Every proposer i asks $1/(1+\delta^2)$ for herself, offers 0 to $(i+1)\bmod 3$, and $\delta^2/(1+\delta^2)$ to $(i+2)\bmod 3$, for all $i = 1, 2, 3$. Every responder j accepts any offer with $a_j \geq \delta^2/(1+\delta^2)$ and rejects otherwise, for $j = 1, 2, 3$. It is easy to verify that this is an equilibrium, as the responder who is offered a positive share has to wait for two rounds before it is her turn to make the offer. This equilibrium is independent of the parameter α that measures altruism.

5.4. Multi-issue bargaining

In Section 3 it was assumed that bargaining outcomes $a \in \Delta$ live in a simplex. This amounts to assuming that there is only a single issue that is negotiated. On the other hand, an important strand of the literature has studied *multi-issue* bargaining in the sense that several surpluses may be distributed (Inderst, 2000; Busch and Horstmann, 2002; In and Serrano, 2003, 2004). The proof of Theorem 1 makes no use of the dimensionality of Δ, though. Since it relies exclusively on continuity and compactness, the simplex Δ could easily be replaced by a cube Δ^K, or any nonempty compact subset of $\mathbb{R}^K$, that captures $K > 1$ distinct issues about which players bargain. The arguments establishing existence of equilibrium remain unchanged.

5.5. Procedural preferences

In the main part of the paper we have taken a non-procedural stance, according to which bargaining partners care only about how the surplus is finally split and about when agreement is reached—but not about how this is brought about. Formally this is expressed by defining utility functions on the space Z of economic outcomes. If the procedure of bargaining in itself – how outrageous or how modest offers are, how stubbornly people behave, whether they enjoy or despise lengthy negotiations, etc. – influences well-being, then preferences need to be defined directly on plays.[6] That is, the appropriate domain for "procedural" preferences is then the set W of plays, rather than the set Z. This would not pose a problem for the existence proof, though. For, the underlying theorem (Alós-Ferrer and Ritzberger, 2016b, Theorem 1; 2016a, Theorem 7.4) holds for preferences defined on plays, even if those are purely ordinal, i.e., not necessarily representable by utility functions. Of course, the main hypothesis, continuity with respect to a (compact separated) topology on the set of plays, would then have to apply to these preferences defined on plays, but, with this modified hypothesis, Theorem 1 can be immediately extended to cover procedural preferences.

5.6. A simple non-stationary example without immediate agreement

The following example illustrates a non-stationary, player-asymmetric environment where there is no immediate agreement.

Example 8. Let $n = 3$ and set $r^t = (1, 2, 3)$ for all t odd, $r^t = (2, 1, 3)$ for all t even. That is, players 1 and 2 make proposals alternatingly, with the other player voting first. Player 3 never gets to propose, and always votes second. For $t = 1, 2$, the voting rule is unanimity, i.e. $\psi^t(b) = 1 \Leftrightarrow b = (1, 1)$. Yet, after $t = 3$ player 3 becomes a dummy player. For all $t \geq 3$, $\psi^t(b) = 1 \Leftrightarrow b_2 = 1$, i.e., the rule becomes unanimity among 1 and 2.

Let $u_i(a, t) = \delta^{t-1} a_i$ for $i = 1, 2$. Since player 3 is effectively out of the game for all $t \geq 3$, it follows that in all subgames at or after $t = 3$ the only subgame perfect equilibrium is as in Example 5. That is, player 1 always proposes $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}, 0\right)$ and accepts if and only if offered a share of at least $\frac{\delta}{1+\delta}$, and symmetrically for player 2.

Suppose that player 3 has other-regarding, spiteful preferences of the form

$$u_3(a, t) = \delta^{t-1} a_3 + \min\{t-1, 2\} \left(1 - \delta^{t-1}(1 - a_3)\right).$$

That is, player 3 values getting a large share for himself (and $u_3(a, t)$ is increasing in a_3 for any fixed t), but also values reducing the discounted joint payoffs of the other two players as much as possible, and the utility weight of this reduction increases every period until $t = 3$ (when the player becomes a dummy). Since u_i is continuous in the sense of Section 3 for all $i = 1, 2, 3$, with $u_1(\infty) = u_2(\infty) = 0$, $u_3(\infty) = 2$, Theorem 1 applies. In any subgame perfect equilibrium, at $t = 2$, players know that if player 2's offer is rejected, the allocation $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}, 0\right)$ will be realized at $t = 3$. Hence, if an offer at $t = 2$ is to be preferred by both players 1 and 2 to waiting until $t = 3$, it must give player 1 at least $\frac{\delta}{1+\delta}$ and player 2 at least $\frac{\delta^2}{1+\delta}$. Hence, no offer at $t = 2$ can be made and accepted if it gives player 3 strictly more than $1 - \frac{\delta}{1+\delta} - \frac{\delta^2}{1+\delta} = 1 - \delta$. For, given such an allocation, either player 2 prefers to wait (and hence will not make the offer if it is to be accepted) or player 1 prefers to wait (and hence will vote against). However, a direct computation shows that if an offer at $t = 2$ gave player 3 $1 - \delta$, that player would vote against it, because he prefers to receive 0 one period later than $1 - \delta$ at $t = 2$. Indeed,

$$u_3(1-\delta, 2) < u_3(0, 3) \Leftrightarrow \delta(1-\delta) + (1 - \delta(\delta)) < 2\left(1 - \delta^2\right)$$
$$\Leftrightarrow \delta(1-\delta) < 1 - \delta^2 b \Leftrightarrow \delta < 1 + \delta$$

and the last inequality holds immediately. It follows that player 3 would also reject any offer at $t = 2$ giving him strictly less than $1 - \delta$. We conclude that, in a subgame perfect equilibrium, the equilibrium offer of player 2 at $t = 2$ must be rejected.

Analogously, in any subgame perfect equilibrium, at $t = 1$ players know that if player 1's offer is rejected, the allocation $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}, 0\right)$ will be realized at $t = 3$. Hence, if an offer at $t = 1$ is to be preferred by both players 1 and 2 to waiting until $t = 3$, it must give player 1 at least $\frac{\delta^2}{1+\delta}$ and player 2 at least $\frac{\delta^3}{1+\delta}$. Hence, no offer at $t = 1$ can be made and accepted if it gives player 3 strictly more than $1 - \frac{\delta^2}{1+\delta} - \frac{\delta^3}{1+\delta} = 1 - \delta^2$. Again, a direct computation shows that player 3 will vote against any offer at $t = 1$ giving him $1 - \delta^2$, since $u_3(1-\delta^2, 1) < u_3(0, 3) \Leftrightarrow (1 - \delta^2) < 2\left(1 - \delta^2\right)$ and the last inequality is obviously true. That is, in all subgame perfect equilibria of this game players 1 and 2 purposefully make offers which will be rejected in the first two periods, and reach an agreement in period 3.

This example is stylized, but it can of course be generalized. The key of the example is the willingness of one player to wait in order to reduce other players' payoffs, a phenomenon allowed by other-regarding preferences, and the feasibility of shutting down this player by waiting until an appropriate environmental change, which is allowed by the non-stationarity of the setting. It also illustrates that the setting allows for the effective set of bargaining agents to be time-dependent.

5.7. A non-stationary environment without stationary equilibria

Stationary equilibrium assumes that players are unresponsive to the past. To bypass a lengthy discussion of what that precisely

[6] For example, Li (2007) studies bilateral bargaining when players prefer an impasse to accepting offers which would give them lower discounted payoffs than those of offers they have previously rejected.

means, consider the archetypal bilateral bargaining model with alternating proposers. Player 1 begins by making an offer to which player 2 responds by either accepting or rejecting. In the former case the proposal is implemented and the game ends; in the latter case the clock turns to the next period and player 2 gets to make an offer. Player 2's proposal may be accepted by player 1, ending the game, or rejected, which puts player 1 back into the proposer role next period.

At least in this simple model any stationary equilibrium must entail a pair of functions $b_i : [0, 1] \to \{0, 1\}$, that specify how players $i = 1, 2$ respond to offers $a \in [0, 1]$, such that these functions are independent of history and time. To exhibit an example where no stationary equilibrium exists, we construct a bilateral bargaining model where such a pair of history-independent response rules cannot exist in equilibrium.

Let $q_t = 1 - 2^{-t}$ for all $t = 1, 2, \ldots$, so $\{q_t\}_{t=1}^{\infty}$ is a strictly increasing sequence with $q_t \in (0, 1)$ for all t and $q_t \to 1$ as $t \to \infty$. Denote by $a \in [0, 1]$ the share of player 1 and define, for each $t = 1, 2, \ldots$, and $i = 1, 2$,

$$u_i(a, t) = \begin{cases} -1/t & \text{if } a \le q_t \\ [3 + 2^{t+2}(a-1)]/t & \text{if } q_t < a < q_{t+1} \\ 1/t & \text{if } q_{t+1} \le a \le q_{t+2} \\ [-3 + 2^{t+4}(1-a)]/t & \text{if } q_{t+2} < a < q_{t+3} \\ -1/t & \text{if } a \ge q_{t+3} \end{cases}$$

Note that both players have the same identical interests, and the stakes vanish as time passes: at period t, $-1/t \le u_i(a, t) \le 1/t$ for all $a \in [0, 1]$. Define also $u_i(\infty) = 0$. It is immediate that these functions are continuous (including continuity at infinity), and hence this example fulfills the assumptions of Theorem 1 (and it is easy to find non-stationary subgame perfect equilibria).

Suppose there exists a subgame-perfect equilibrium with stationary response rules b_1, b_2. Consider any $a \in [0, 1)$. Since $q_t \to 1$, there exists t even such that $a < q_t$. In the subgame where player 2 proposes a at time t, player 1 must reject it, because acceptance yields payoff $-1/t$, but rejection leads to a payoff of at least $-1/(t+1)$. Hence $b_1(a) = 0$. Since $t+1$ is odd and $a < q_{t+1}$, it follows analogously that also $b_2(a) = 0$. Further, $a = 1$ yields payoff -1 for player 1 at $t = 1$ and payoff $-1/2$ for player 2 at $t = 2$, while rejection leads to payoffs of at least $-1/2$ and $-1/3$, respectively. Hence $b_1(1) = b_2(1) = 0$ also holds.

We conclude that the response rules must specify universal rejection, and hence the equilibrium outcome is eternal disagreement with payoff $u_i(\infty) = 0$. But this cannot be a subgame perfect equilibrium, because in any subgame starting at period t with an offer $a \in (q_{t+1}, q_{t+2})$, the responder obtains a strictly positive payoff by accepting. Thus, this game has no stationary subgame perfect equilibrium.

6. Conclusions

This paper has studied a general class of multi-lateral bargaining models. No particular model of impatience is required, hence allowing for geometric or hyperbolic discounting, externalities and other-regarding preferences are allowed, and bargaining protocols can be arbitrary and vary over time—hence, no stationary structure is assumed. The main insight is that all games in that class have a subgame perfect equilibrium, provided the utility functions are continuous. Continuity here includes the possibility of perpetual disagreement, that is, it also holds "at infinity". This result hence eliminates the necessity to demonstrate existence of equilibrium for each bargaining problem separately, by providing a general existence proof for a wide class of problems.

Appendix A. Topologies

A.1. The space of economic outcomes

A bargaining outcome is an allocation and the time at which it is accepted, $(a, t) \in \Delta \times \{1, 2, \ldots\}$, or perpetual disagreement, which is represented by the symbol ∞. Hence, the space of outcomes is

$$Z = (\Delta \times \{1, 2, \ldots\}) \cup \{\infty\}.$$

To endow it with a natural topology, we will view it as a quotient set of the larger, instrumental space

$$\bar{Z} = \Delta \times \{1, 2, \ldots, \infty\}.$$

This latter space has a product structure and is naturally well-behaved if the appropriate topologies are taken for the factors. Endow the space of allocations Δ with the Euclidean topology, which makes Δ compact and Hausdorff. The space $\{1, 2, \ldots, \infty\}$ is also compact and Hausdorff if it is endowed with the one-point compactification of the discrete topology (also known as the Fort topology with distinguished point ∞; see, e.g., Steen and Seebach, 1978, p. 52), that is, a subset of $\{1, 2, \ldots, \infty\}$ is open if and only if its complement is either finite or contains ∞. This topology is just the discrete topology (all sets are open) if restricted to $\{1, 2, \ldots\}$, but it requires open sets containing ∞ to contain some terminal segment $\{t, t+1, \ldots, \infty\}$. It will be shown below that this topology is closely related to notions of "continuity at infinity".

The space $\bar{Z}$ is endowed with the product topology and is therefore compact and Hausdorff. Consider the projection $\pi : \bar{Z} \to Z$ given by $\pi(a, t) = (a, t)$ if $t \neq \infty$ and $\pi(a, \infty) = \infty$ for all $a \in \Delta$. This amounts to identifying all outcomes that are sufficiently far in the future to a single equivalence class. Thereby the space Z can be seen as the quotient space $\bar{Z}/(\Delta \times \{\infty\})$ and endowed with the quotient topology (Steen and Seebach, 1978, p. 9), that is, a set $V \subseteq Z$ is open in Z if and only if $\pi^{-1}(V)$ is open in the product topology on $\bar{Z}$.

Lemma A.1. *The set Z endowed with the quotient topology is compact and Hausdorff.*

Proof. Since $\bar{Z}$ is compact, the quotient set Z is also compact. To see that it is also Hausdorff, let $x, y \in \bar{Z}$ with $x \neq \infty \neq y$. Since $\bar{Z}$ is Hausdorff, there exist U_x, U_y open sets in $\bar{Z}$ such that $x \in U_x, y \in U_y$, and $U_x \cap U_y = \emptyset$. Since $\{1, 2, \ldots\}$ is open in $\{1, 2, \ldots, \infty\}$ and $x \neq \infty \neq y$, the sets $U'_x = U_x \cap \Delta \times \{1, 2, \ldots\}$ and $U'_y = U_y \cap \Delta \times \{1, 2, \ldots\}$ are also open sets in $\bar{Z}$ such that $x \in U'_x, y \in U'_y$, and $U'_x \cap U'_y = \emptyset$. Consider the (singleton) classes $[x]$ and $[y]$ in Z. Then $\pi^{-1}(U'_x)$ and $\pi^{-1}(U'_y)$ are open sets of Z with $[x] \in \pi^{-1}(U'_x)$, $[y] \in \pi^{-1}(U'_y)$, and $\pi^{-1}(U'_x) \cap \pi^{-1}(U'_y) = \emptyset$.

Hence all classes $[x]$, $[y]$ different from $[\infty]$ can be separated by open sets. To complete the proof, let $x = (a, t) \in \bar{Z}$. We have to show that the classes $[x]$ and $[\infty]$ can be separated by open sets in the quotient space. To see this, let $U_x = \Delta \times \{1, \ldots, t\}$ and $U_\infty = \Delta \times \{t+1, \ldots, \infty\}$. These sets are disjoint and open in $\bar{Z}$, and fulfill $x \in U_x$ and $\infty \in U_\infty$. Then $\pi^{-1}(U_x)$ and $\pi^{-1}(U_\infty)$ are open sets of Z with $[x] \in \pi^{-1}(U_x)$, $[\infty] \in \pi^{-1}(U_\infty)$, and $\pi^{-1}(U_x) \cap \pi^{-1}(U_\infty) = \emptyset$. □

Abusing notation, denote the class $[x] = \{x\}$ in Z simply by x for each $x \neq \infty$, and the class $[\infty] = \Delta \times \{\infty\}$ by ∞. For convenience, if $V \subseteq Z$, the set $V \setminus \{\infty\}$ can be viewed as a subset of $\bar{Z}$ and hence cumbersome notation can be avoided. In particular, if $\infty \notin V$ the set V can be seen both as a subset of Z and as a subset of $\bar{Z}$. If $\infty \in V \subseteq Z$, the set $\pi^{-1}(V) \subseteq \bar{Z}$ corresponds to V with the addition of all pairs of the form (a, ∞). Formally,

$$\pi^{-1}(V) = \begin{cases} V & \text{if } \infty \notin V \\ (V \setminus \{\infty\}) \cup (\Delta \times \{\infty\}) & \text{if } \infty \in V. \end{cases}$$

That is, if an open set $V \subseteq Z$ contains ∞, the open set $\pi^{-1}(V) \subseteq \bar{Z}$ contains the whole "limit slice" $\Delta \times \{\infty\}$. The following lemma identifies an important property of such sets, which will be used in the proofs below.

Lemma A.2. *Let $U \subseteq \bar{Z}$ be an open set in the product topology on $\bar{Z}$ such that $\Delta \times \{\infty\} \subseteq U$. Then, there exists $T \geq 1$ such that $\Delta \times \{T, T+1, \ldots, \infty\} \subseteq U$.*

Proof. For each $a \in \Delta$, $(a, \infty) \in U$. Since U is open, there exist $U_1(a)$ open in Δ and $U_2(a)$ open in $\{1, 2, \ldots, \infty\}$ such that $(a, \infty) \in U_1(a) \times U_2(a) \subseteq U$.

The sets $\{U_1(a) \mid a \in \Delta\}$ form an open cover of Δ. Since Δ with the Euclidean topology is compact, there exists a finite subcover, i.e. $a_1, \ldots, a_m \in \Delta$ such that $U_1(a_1) \cup U_1(a_2) \cup \cdots \cup U_1(a_m) = \Delta$. For each $j = 1, \ldots, m$, the set $U_2(a_j)$ is open in $\{1, 2, \ldots, \infty\}$ but contains ∞. Open sets in the topology of this space are those whose complements are either finite or contain ∞. Since the former cannot happen, it follows that for each $j = 1, 2, \ldots, m$, there exists t_j such that $\{t_j, t_j + 1, \ldots, \infty\} \subseteq U_2(a_j)$. Let $T = \max_{j=1,\ldots,m} t_j$. Then, $U_1(a_j) \times \{T, T+1, \ldots, \infty\} \subseteq U$ for all j and, since the sets $U_1(a_j)$ cover Δ, it follows that $\Delta \times \{T, T+1, \ldots, \infty\} \subseteq U$. □

A.2. The space of plays

The space of plays is given by

$$W = \left(\bigcup_{T=1}^{\infty} W^T \right) \cup W^\infty$$

and the map from plays to bargaining outcomes by $\varphi : W \to Z$ with

$$\varphi(w) = \begin{cases} \left(a^T, T\right) & \text{if } w = \left(a^t, b^t\right)_{t=1}^{T} \in W^T,\ T < \infty \\ \infty & \text{if } w \in W^\infty \end{cases} \tag{1}$$

Viewed as an infinite union the space W does not lend itself to a natural well-behaved topology. Therefore, we will embed it into an instrumental space of sequences from an enlarged space of instantaneous actions. The latter can be endowed with a natural well-behaved product topology, but does contain many situations which cannot be interpreted in terms of the game. Thus, the actual space of plays will be a proper subset of the set of sequences. This subset will be shown to be closed in the product topology. In this way, the relative topology (inherited by the closed subset) will provide a well-behaved topology for the space of plays.

Expand the set of allocations Δ with a dummy action $*$, indicating that agreement was achieved in the past. Likewise, expand the set of voting profiles B with a dummy profile $**$, indicating that no voting is called for, as the game has ended. Take the set of instantaneous action profiles as

$$\Omega = (\Delta \cup \{*\}) \times (B \cup \{**\}).$$

Endow Δ with the Euclidean topology and $\Delta \cup \{*\}$ with a Euclidean-expanded topology where $V \subseteq \Delta \cup \{*\}$ is open if and only if $V \cap \Delta$ is open in Δ. That is, the topology on $\Delta \cup \{*\}$ simply "ignores" the additional point, and the singleton $\{*\}$ is declared open. This topology makes $\Delta \cup \{*\}$ compact and Hausdorff. The finite set $B \cup \{**\}$ is endowed with the discrete topology, which of course makes it also compact and Hausdorff. The set $\Omega = (\Delta \cup \{*\}) \times (B \cup \{**\})$ is endowed with the product topology and hence it is also compact and Hausdorff. Then consider the infinite product

$$\Omega^\infty = [(\Delta \cup \{*\}) \times (B \cup \{**\})]^\infty$$

endowed with the product topology. This space is compact and Hausdorff, but it has no natural correspondence with the space of plays. For instance, sequences where $a \in \Delta$ appears strictly after an occurrence of $*$ cannot be a play. The following identifies the sequences in Ω^∞ that correspond to plays.

Definition A.1. A sequence $\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{\infty} \in \Omega^\infty$ is a *bargaining sequence* if

(S.1) $\bar{a}^1 \in \Delta$;
(S.2) $\bar{a}^t = * \Leftrightarrow \bar{b}^t = **$, for all $t = 1, 2 \ldots$;
(S.3) $\left(\bar{a}^t, \bar{b}^t\right) \in \Delta \times B \Leftrightarrow \psi^s\left(\bar{b}^s\right) = 0\ \forall s = 1, \ldots, t-1$ for all $t = 1, 2, \ldots$

Condition (S.1) states that bargaining begins with an offer. (S.2) demands that dummy options always occur in both coordinates of the instantaneous action profile. (S.3) stipulates that after rejections bargaining continues and after an acceptance it ends. Let $\bar{W} \subset \Omega^\infty$ denote all bargaining sequences.

Lemma A.3. *The set $\bar{W}$ is closed in Ω^∞.*

Proof. It will be proved that the complement $\Omega^\infty \setminus \bar{W}$ of $\bar{W}$ is open (in the product topology on Ω^∞) by showing that it contains an open neighborhood for all its points. More precisely, for any sequence $\bar{w} \in \Omega^\infty \setminus \bar{W}$ and every possible violation of (S.1–3) we will exhibit a basic open set U that contains $\bar{w}$ and is contained in $\Omega^\infty \setminus \bar{W}$.[7] Let $\bar{w} = \left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{\infty} \in \Omega^\infty \setminus \bar{W}$. Then $\bar{w}$ violates at least one of the conditions from Definition A.1.

If $\bar{w}$ fails (S.1), i.e. $\bar{a}^1 = *$, take $U = (\{*\} \times (B \cup \{**\})) \times \Omega^\infty$. If $\bar{w}$ violates (S.2), then there is $t \geq 2$ such that either $\bar{a}^t = *$ and $\bar{b}^t \in B$ or $\bar{a}^t \in \Delta$ and $\bar{b}^t = **$. In the first case let $U = \Omega^{t-1} \times (\{*\} \times B) \times \Omega^\infty$, in the second case let $U = \Omega^{t-1} \times (\Delta \times \{**\}) \times \Omega^\infty$.

If $\bar{w}$ fails the "if"-part of (S.3) with $\bar{a}^t \in \Delta$ or its "only if"-part, then there is t such that either $\psi^s(\bar{b}^s) = 0$ for all $s = 1, \ldots, t-1$ but $\bar{b}^t = **$ or $\left(\bar{a}^t, \bar{b}^t\right) \in \Delta \times B$ but $\psi^s(\bar{b}^s) = 1$ for some $s = 1, \ldots, t-1$. For these cases let $U = \left(\times_{s=1}^{t}\left((\Delta \cup \{*\}) \times \{\bar{b}^s\}\right)\right) \times \Omega^\infty$. Finally, if $\bar{w}$ violates the "if"-part of (S.3) with $\bar{a}^t = *$, let

$$U = \left(\times_{s=1}^{t-1}\left((\Delta \cup \{*\}) \times \{\bar{b}^s\}\right)\right) \times (\{*\} \times (B \cup \{**\})) \times \Omega^\infty.$$

Since it is not difficult to check that all the sets U are basic open neighborhoods of $\bar{w}$ contained in $\Omega \setminus \bar{W}$, the proof is complete. □

Since $\bar{W}$ is closed in Ω^∞ and the latter space is compact and Hausdorff with the product topology, the set $\bar{W}$ together with the relative topology inherited from Ω^∞ (as a subspace) is compact and Hausdorff. All that remains is to show that there is a natural bijection between $\bar{W}$ and the space of plays W. Let $\Lambda : W \to \bar{W}$ be given by

$$\Lambda(w) = \begin{cases} \left(a^1, b^1, \ldots, a^T, b^T,\right. & \\ \left. *, **, *, **, \ldots\right) & \text{if } w = \left(a^t, b^t\right)_{t=1}^{T} \in W^T,\ T < \infty \\ w & \text{if } w \in W^\infty. \end{cases}$$

This mapping is obviously injective (one to one). To show that is surjective (onto), let $\bar{w} \in \bar{W}$. If $\bar{w} \in (\Delta \times B)^\infty$, then $\bar{w} \in W^\infty$

[7] A basis for the product topology consists of the product sets for which all but finitely many of the factors are unrestricted. An open set from this basis is a *basic open set*.

and $\Lambda(\bar{w}) = \bar{w}$. If $\bar{w} = (\bar{a}^t, \bar{b}^t)_{t=1}^{\infty} \notin (\Delta \times B)^{\infty}$, by (S.3) there is some $t \geq 1$ such that $\bar{a}^t = *$. Let $\bar{t} = \min\{t \mid \bar{a}^t = *\}$. By (S.3) again, $\bar{b}^s \neq **$, for all $s < \bar{t}$. By (S.2) and (S.3), $\psi^s(\bar{b}^s) = 0$ for all $s < \bar{t} - 1$. By (S.3) again, $\psi^s(\bar{b}^{\bar{t}-1}) = 1$. It follows that $(\bar{a}^t, \bar{b}^t)_{t=1}^{\bar{t}-1} \in W^{\bar{t}-1}$. Further, by (S.2) $\bar{a}^s = *$ and $\bar{b}^s = **$ for all $s \geq \bar{t}$, hence $\Lambda\left((\bar{a}^t, \bar{b}^t)_{t=1}^{\bar{t}-1}\right) = \bar{w}$.

Hence, Λ is a bijection between W and $\bar{W}$. Formally, the topology on $\bar{W}$ defines a compact Hausdorff topology on W by declaring $U \subseteq W$ to be open if and only if $\Lambda(U)$ is open in $\bar{W}$. In practice, we can simply identify the spaces W and $\bar{W}$ and rely on the topology on $\bar{W}$. To economize on notation, in the sequel we will follow this approach and identify W and $\bar{W}$, meaning that a play $w \in W$ can also be seen as the corresponding bargaining sequence in the sense of Definition A.1.

A.3. The outcome mapping

Having endowed the set W of plays and the set Z of outcomes with compact Hausdorff topologies, the next step is to show that the mapping φ from (1), as a mapping between these topological spaces, is continuous.

Lemma A.4. *The outcome mapping $\varphi : W \to Z$ is continuous.*

Proof. Let $V \subseteq Z$ be an open set. By construction of the topology on Z, as a quotient topology derived from $\bar{Z} = \Delta \times \{1, 2, \ldots, \infty\}$, the subset $\pi^{-1}(V)$ of $\bar{Z}$ is open in the product topology on $\bar{Z}$. To prove that $\varphi^{-1}(V)$ is open in W, it will be shown that $\varphi^{-1}(V)$ contains an open neighborhood for each of its elements. Let $w \in \varphi^{-1}(V)$ and let $v = \varphi(w) \in V$. Two cases need to be distinguished.

First, suppose that $v \neq \infty$. In this case $\pi^{-1}(v) = \{v\}$. Then, $w \in W^T$ and $v = (a^T, T)$ for some finite $T \geq 1$ and $a^T \in \Delta$. Denoting $w = \left(\tilde{a}^t, \tilde{b}^t\right)_{t=1}^{T}$, it also follows that $\tilde{a}^T = a^T$ and $\psi^T(\tilde{b}^T) = 1$. Since V is open in $\bar{Z}$ and $v \in \pi^{-1}(V)$, there are U_1 open in Δ and U_2 open in $\{1, 2, \ldots, \infty\}$ such that $v = (a^T, T) \in U_1 \times U_2 \subseteq \pi^{-1}(V)$. Then the set

$$U = (\Delta \times B)^{T-1} \times \left(U_1 \times \{\tilde{b}^T\}\right) \times \Omega^{\infty}.$$

is an open basic set of the product topology on Ω^{∞}. Hence, $U \cap \bar{W}$ is an open set of $\bar{W}$ and, through the identification of this set with the space of plays, of W. For every $w' \in U \cap \bar{W}$, since $\psi^T(\tilde{b}^T) = 1$, it follows that $\pi^{-1}(\varphi(w')) = \{\varphi(w')\} \subseteq U_1 \times \{T\} \subseteq U_1 \times U_2 \subseteq \pi^{-1}(V)$. Therefore, $w \in U \cap \bar{W} \subseteq \varphi^{-1}(V)$.

Suppose now that $v = \infty$, that is, w is a perpetual disagreement path. In this case, $\pi^{-1}(v) = \Delta \times \{\infty\} \subseteq \pi^{-1}(V)$. By Lemma A.2, there is $T \geq 1$ such that $\Delta \times \{T, T+1, \ldots, \infty\} \subseteq \pi^{-1}(V)$. Then the set $U = (\Delta \times B)^T \times (\Omega)^{\infty}$ is an open basic set of the product topology on Ω^{∞}, $U \cap \bar{W}$ is an open set of $\bar{W}$, and – by the identification of this set with the space of plays – of W. For every $w' \in U \cap \bar{W}$, either agreement occurs at a finite time larger than or equal to T, in which case $\pi^{-1}(\varphi(w')) = \{\varphi(w')\}$, or there is perpetual disagreement, $\varphi(w') = \infty$. In both cases, $\pi^{-1}(\varphi(w')) \subseteq \Delta \times \{T, T+1, \ldots, \infty\} \subseteq \pi^{-1}(V)$. Hence $w \in U \cap \bar{W} \subseteq \varphi^{-1}(V)$. □

Appendix B. The bargaining tree

Following the original approach of von Neumann and Morgenstern (1944), a node in an extensive form game is simply a collection of outcomes, those that are still available when a player decides at that node. A node precedes another node if and only if the latter is properly contained in the former. Intuitively, decisions discard possible outcomes and hence reduce the size of the nodes.[8] For the perfect information case treated here, the game is fully characterized by the collection of nodes (the "tree") and the specification of which player is active when.

The nodes in the tree of the bargaining game are as follows. At each round $s = 1, 2, \ldots$ there are n "slices", $(s, 1)$ to (s, n), where the proposer and the second-to-last responder act, respectively. At slice $(s, 1)$, the player selected as proposer in that round (which may vary depending on previous votes) makes an offer. The nodes at this slice are of the form

$$x_i^s\left(\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{s-1}\right) = \bigcup_{T \in \{s,\ldots,\infty\}} \left\{\left(a^t, b^t\right)_{t=1}^{T} \in W^T \;\middle|\; a^t = \bar{a}^t,\ b^t = \bar{b}^t\ \forall t \leq s-1\right\}$$

where $i = r_1^s(\bar{b}^1, \ldots, \bar{b}^{s-1})$ is the proposer (who acts at this node), for each $\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{s-1}$ describing a previous history of rejected offers, that is, fulfilling $\psi\left(\bar{b}^t\right) = 0$ for all $t = 1, \ldots, s-1$. (The subindex in the union describing the node x_i^s runs from s to ∞; in particular, $T = \infty$ is allowed to capture plays with perpetual disagreement.) Hence, the set of non-terminal nodes at slice $(s, 1)$ is given by

$$X_1^s = \left\{ x_{i,1}^s\left(\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{s-1}\right) \;\middle|\; \begin{array}{l} \left(\bar{a}^t, \bar{b}^t\right) \in \Delta \times B \text{ and} \\ \psi\left(\bar{b}^t\right) = 0\ \forall t \leq s-1 \\ i = r_1^s(\bar{b}^1, \ldots, \bar{b}^{s-1}) \end{array} \right\}.$$

In particular, at slice $(1, 1)$ the set X_1^1 consists of only one node, containing all plays, which is, as a set, identical with W—the root of the game.

For every $k = 2, \ldots, N$, the slice (s, k) consists of nodes of the form

$$x_{j,k}^s\left(\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{s-1}, \bar{a}^s, \left(\bar{b}_\ell^s\right)_{\ell=2}^{k-1}\right) = \bigcup_{T \in \{s,\ldots,\infty\}} \left\{\left(a^t, b^t\right)_{t=1}^{T} \in W^T \;\middle|\; \begin{array}{l} a^t = \bar{a}^t\ \forall t \leq s, \\ b^t = \bar{b}^t\ \forall t \leq s-1, \\ b_\ell^s = \bar{b}_\ell^s\ \forall \ell = 2, \ldots, k-1 \end{array} \right\}$$

following a sequence of rejected offers, where $j = r_k^s(\bar{b}^1, \ldots, \bar{b}^{s-1})$ is the $(k-1)$th responder, who plays at this node and whose identity depends on previous votes. That is, the set of nodes at slice (s, k) with $k \geq 2$ is as given in Box I below.

Finally, the set of terminal nodes after acceptance of offers at round s, which formally belong to slice $(s+1, 1)$, is given by the singletons of the corresponding finite plays, $E^{s+1} = \{\{w\} \mid w \in W^s\}$ for each $s = 1, 2, \ldots$

Following the notation in Alós-Ferrer and Ritzberger (2016a; 2016b), the slice (s, k), viewed as a collection of nodes, is denoted by $Y_{s,k}$. Thus, $Y_{1,1} = \{W\}$ and $Y_{s,1} = X_1^s \cup E^s$ for each $s \geq 1$. Since terminal nodes only occur after all votes have been cast, also $Y_{s,k} = X_k^s$ for each $k \geq 2$.

B.1. Properties of the bargaining tree

Here we show that the tree as defined fulfills the hypotheses for the result of Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4).

B.2. Well-behaved

The first condition stipulates that the game is "well-behaved". This condition has two parts. The first is that the preferences of all players are continuous with respect to the topology on the space

[8] This approach has been developed for arbitrarily large extensive form games (see Alós-Ferrer and Ritzberger, 2016a), showing e.g. the relation with popular "graphical approaches" where trees are viewed as graphs.

$$X_k^s = \left\{ x_{j,k}^s\left(\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{s-1}, \bar{a}^s, \left(\bar{b}_\ell^s\right)_{\ell=2}^{k-1}\right) \;\middle|\; \begin{array}{l} \bar{a}^t \in \Delta\ \forall t \leq s, \\ \bar{b}^t \in B\ \forall t \leq s-1, \\ \bar{b}_\ell^s \in \{0,1\}\ \forall \ell = 2,\ldots,k-1, \\ \psi\left(\bar{b}^t\right) = 0\ \forall t = 1,\ldots,s-1 \\ j = r_k^s(\bar{b}^1,\ldots,\bar{b}^{s-1}) \end{array} \right\}.$$

Box I.

of plays. Since here payoffs are defined on bargaining outcomes, $u_i : Z \to \mathbb{R}$, plays need to be mapped into outcomes by the function $\varphi : W \to Z$. As φ is continuous by Lemma A.4, the composition $u_i \circ \varphi$ is continuous if $u_i : Z \to \mathbb{R}$ is continuous. The latter amounts to continuity with respect to the topology on Z derived as the quotient topology of $\bar{Z}$ with respect to $\Delta \times \{\infty\}$. The following shows that our continuity assumptions on u_i are exactly equivalent to continuity with respect to this topology. Note that (ii) below amounts to "continuity at infinity".

Proposition B.1. *The function $u_i : Z \to \mathbb{R}$ is continuous with respect to the topology on Z derived as the quotient topology of $\bar{Z}$ with respect to $\Delta \times \{\infty\}$ if and only if it is continuous in the sense of Definition 1, that is,*

(i) for each $t = 1, 2, \ldots$, the function $u_i^t : \Delta \to \mathbb{R}$ given by $u_i^t(a) = u_i(a,t)$ for each $a \in \Delta$ is continuous (with respect to the Euclidean topologies on Δ and $\mathbb{R}$), and

(ii) for each $\varepsilon > 0$ there exists $T \in \{1, 2, \ldots\}$ such that $|u_i(a,t) - u_i(\infty)| < \varepsilon$ for all $a \in \Delta$ and all $t \geq T$.

Proof. "if:" Suppose that (i) and (ii) hold. To see that u_i is continuous, let $V \subseteq \mathbb{R}$ be an open set of real numbers. To prove that $u_i^{-1}(V)$ is open in Z, we will show that $u_i^{-1}(V)$ contains an open neighborhood for each for its elements. To see this, let $z \in u_i^{-1}(V)$ and distinguish two cases.

First, if $z \neq \infty$, then $z = (a,t)$ for some $a \in \Delta$ and $t \in \{1, 2, \ldots\}$. Thus, $u_i^t(a) = u_i(a,z) \in V$ and by continuity $\left(u_i^t\right)^{-1}(V)$ is open in Z. By construction of the topology on Z, $\left(u_i^t\right)^{-1}(V) \times \{t\}$ is open in Z such that $z \in \left(u_i^t\right)^{-1}(V) \times \{t\} \subseteq u_i^{-1}(V)$. Second, suppose $z = \infty$. Since V is open and $u_i(\infty) \in V$, there is $\varepsilon > 0$ such that $(u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon) \subseteq V$. By (ii), there exists $T \in \{1, 2, \ldots\}$ such that for each $a \in \Delta$ and each $t \geq T$, $|u_i(a,t) - u_i(\infty)| < \varepsilon$. It follows that $\{\infty\} \cup (\Delta \times \{1, 2, \ldots\}) \subseteq u_i^{-1}((u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon)) \subseteq u_i^{-1}(V)$. As $\{\infty\} \cup (\Delta \times \{1, 2, \ldots\}) = \pi(\Delta \times \{1, 2, \ldots, \infty\})$, this set is an open neighborhood of ∞ in Z.

"only if:" Let $u_i : Z \to \mathbb{R}$ be continuous. To see (i), fix $t \in \{1, 2, \ldots\}$ and let $V \subseteq \mathbb{R}$ be an open set of real numbers. Then,

$$\left(u_i^t\right)^{-1}(V) = \{a \in \Delta \mid u_i(a,t) \in V\} = u_i^{-1}(V) \cap (\Delta \times \{t\}).$$

The set $u_i^{-1}(V)$ is open in Z because u_i is continuous, and the set $\Delta \times \{t\}$ is open by construction of the topology on Z, since $\pi^{-1}(\Delta \times \{t\}) = \Delta \times \{t\}$. Hence, $\left(u_i^t\right)^{-1}(V)$ is open in Z. Since V was arbitrary, u_i^t is continuous.

To see (ii), let $\varepsilon > 0$. As $\infty \in Z$, $u_i(\infty) \in \mathbb{R}$ and the interval $(u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon)$ is open in $\mathbb{R}$. By continuity $u_i^{-1}((u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon))$ is open in Z and contains ∞. By Lemma A.2, there is $T \geq 1$ such that

$$\Delta \times \{T, T+1, \ldots, \infty\} \subseteq \pi^{-1}\left(u_i^{-1}((u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon))\right).$$

Thus, for all $t \geq T$ and all $a \in \Delta$, $(a,t) = \pi^{-1}(a,t)$ fulfills $u_i(a,t) \in (u_i(\infty) - \varepsilon, u_i(\infty) + \varepsilon)$, establishing (ii). □

The second part of well-behavedness demands that the non-terminal nodes at each slice $Y_{s,k}$ are partitioned into finitely many cells,[9] each of which consists of decision points of a single player, such that the set of plays passing through each cell is closed relative to the plays passing through the slice (the set $W(Y_{s,k})$ of plays).

Begin with slice $Y_{s,1}$ for $s \geq 1$. For $s = 1$ the condition is trivially true, because at the root a single player moves. For $s > 1$ and a given player $i \in I$ who proposes at slice $(s,1)$, choose a play $\bar{w} = \left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{T} \in W\left(Y_{s,1}\right)$ (where T may be ∞), which does *not* pass through any decision point of i at $Y_{s,1}$. Then $U = \times_{t=1}^{s-1}\left(\Delta \times \{\bar{b}^t\}\right) \times \Omega^\infty$ is a basic open set in Ω^∞, hence open in $\bar{W}$ and, by the identification of plays with bargaining sequences, open in W. And U contains $\bar{w}$. Further, no play in U can pass through a decision point of i at $Y_{s,1}$, because if it would, last round's voting profile $\bar{b}^{s-1}$ would have given her the move also along $\bar{w}$. Thus, $U \cap W\left(Y_{s,1}\right)$ is relatively open in $W\left(Y_{s,1}\right)$, contains $\bar{w}$, and is contained in the complement of i's decision points at $Y_{s,1}$. As $\bar{w} \in W\left(Y_{s,1}\right)$ was arbitrary, i's cell at $Y_{s,1}$ is closed.

Next, for $s \geq 1$ and $k > 1$ at slice $Y_{s,k}$ and a given player i, who moves at $Y_{s,k}$, choose a play $\bar{w} = \left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{T} \in W(Y_{s,k})$ (where T may be ∞), which does *not* pass through any decision point of i at $Y_{s,k}$. Then $U = \times_{t=1}^{s}\left(\Delta \times \{\bar{b}^t\}\right) \times \Omega^\infty$ is a basic open set in Ω^∞, hence open in $\bar{W}$ and, by the identification of plays and bargaining sequences, open in W. As $\bar{w} \in U$ but i does not move at $Y_{s,k}$ along any member of U, the relatively open set $U \cap W(Y_{s,k})$ is contained in the complement of i's decision points at $Y_{s,k}$ and forms a relative neighborhood of $\bar{w}$. Therefore, i's cell at $Y_{s,k}$ is closed.

Putting the above together completes the verification that the first necessary condition holds, i.e., the bargaining game is indeed well-behaved.

B.3. Closed nodes

The second condition is that all nodes, viewed as sets of plays, are closed sets in the topology on the set of plays W.

Proposition B.2. *All nodes of the bargaining tree are closed sets in the topology on plays.*

Proof. Since the space of plays is Hausdorff, all singleton sets (terminal nodes) are closed. Consider the nodes of a proposer i in a slice s, that is, the nodes in the set X_1^s where player i makes the proposal. Let $x = x_i^s\left(\left(\bar{a}^t, \bar{b}^t\right)_{t=1}^{s-1}\right)$ be one such node. To see that $W \setminus x$ is open, let $w = \left(a^t, b^t\right)_{t=1}^{\infty} \in W \setminus x$ (viewed as a bargaining sequence). There exists some $\hat{t} \in \{1, \ldots, s-1\}$ such that either $a^{\hat{t}} \neq \bar{a}^{\hat{t}}$ or $b^{\hat{t}} \neq \bar{b}^{\hat{t}}$. Suppose the first holds (the second case is analogous). If $a^{\hat{t}} \neq *$, since Δ is Hausdorff, there exists an open set V of Δ such that $a^{\hat{t}} \in V$ but $\bar{a}^{\hat{t}} \notin V$. If $a^{\hat{t}} = *$, let $V = \{*\}$.

9 The sets that together form a partition are referred to as *cells*.

The set $U = (\Omega)^{\hat{t}-1} \times (V \times B) \times (\Omega)^{\infty}$ is an open basic set of Ω^{∞}, hence $U \cap \bar{W}$ is open in $\bar{W}$ (and W). Obviously, $w \in U \cap \bar{W} \subseteq W \backslash x$. The proof for responders' nodes at a given slice is analogous. □

B.4. Open predecessors

The last condition, called the "open predecessors condition", requires that the predecessor mapping is open. For a node x denote by $p(x)$ its immediate predecessor. Then the condition demands that for every slice $Y_{s,k}$ and every set of nodes $V \subseteq Y_{s,k}$, if the set of plays $W(V)$ is open in the relative topology on $W(Y_{s,k})$, then the set of plays passing through the predecessors of nodes in V, $W(p(V))$, is open in the relative topology of the *previous* slice, i.e., on the set $W(Y_{s,k-1})$ if $k \geq 2$ or on the set $W(Y_{s-1,N})$ if $k = 1$.

Proposition B.3. *The tree fulfills the open predecessors condition.*

Proof. Let $V \subseteq Y_{s,k}$ be such that the set of plays $W(V)$ is open in the relative topology on $W(Y_{s,k})$. There are three cases to be distinguished.

Case 1: From $Y_{s,k}$ to $Y_{s,k-1}$ ($k \geq 3$).

In this case $W(Y_{s,k}) = W(Y_{s,k-1})$. Let $w = \left(a^t, b^t\right)_{t=1}^{\infty}$ be a play (viewed as a bargaining sequence) with $w \in W(p(V)) \subseteq W(Y_{s,k-1})$. There exists a node $x \in V$ such that $w \in p(x) = x^s_{i,k-1}\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s, \left(b^s_\ell\right)_{\ell=2}^{k-1}\right)$, where i is the player active at $p(x)$. In fact, then $x = x^s_{j,k}\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s, \left(b^s_\ell\right)_{\ell=2}^{k}\right)$ for some $j \in I$. Let $\bar{w} \in x \subseteq W(V)$. Then the first coordinates of $\bar{w}$ agree with $\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s, \left(b^s_\ell\right)_{\ell=2}^{k}\right)$. Since $W(V)$ is open, it follows that there are open sets of Ω, $U^t \times C^t$, $t = 1, \ldots, s$, such that

$$\bar{w} \in \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times \left(U^s \times \left\{\bar{b} \in B \;\middle|\; \bar{b}^s_\ell = b^s_\ell \;\forall \ell = 2, \ldots, k\right\}\right) \times \Omega^{\infty}\right] \cap \bar{W}$$

and the latter set is contained in $W(V)$. Note that both w and $\bar{w}$ agree with the sequence $\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s, \left(b^s_\ell\right)_{\ell=2}^{k-1}\right)$. Consider the open basic set of Ω^{∞} given by

$$U = \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times \left(U^s \times \left\{\bar{b} \in B \;\middle|\; \bar{b}^s_\ell = b^s_\ell \;\forall \ell = 2, \ldots, k-1\right\}\right) \times (\Omega)^{\infty}\right]$$

Then $U \cap \bar{W}$ is open in $\bar{W}$, hence in W, with $w \in U \cap \bar{W} \subseteq W(p(V))$, demonstrating that the latter set is open.

Case 2: From $Y_{s,2}$ to $Y_{s,1}$ ($k = 2$).

In this case $W(Y_{s,2}) = W(Y_{s,1})$. Let $w = \left(a^t, b^t\right)_{t=1}^{\infty}$ be a play with $w \in W(p(V)) \subseteq W(Y_{s,1})$. There exists a node $x \in V$ such that $w \in p(x) = x^s_{i,1}\left(\left(a^t, b^t\right)_{t=1}^{s-1}\right)$, where i is the proposer active at $p(x)$. Actually, then $x = x^s_{j,2}\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s\right)$, where j is the first responder after the node x.

Let $\bar{w} \in x \subseteq W(V)$. Then the first coordinates of $\bar{w}$ agree with $\left(\left(a^t, b^t\right)_{t=1}^{s-1}, a^s\right)$. Since $W(V)$ is open, there are open sets of Ω, $U^t \times C^t$, $t = 1, \ldots, s$, such that $\bar{w} \in \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times (U^s \times B) \times (\Omega)^{\infty}\right] \cap \bar{W} \subseteq W(V)$. Note that both w and $\bar{w}$ agree with $\left(\left(a^t, b^t\right)_{t=1}^{s-1}\right)$. Consider the open basic set of Ω^{∞} given by $U = \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times (\Omega)^{\infty}\right]$. It follows that $U \cap \bar{W}$ is open in $\bar{W}$, hence in W, with $w \in U \cap \bar{W} \subseteq W(p(V))$, so the latter set is open.

Case 3: From $Y_{s,1}$ to $Y_{s-1,N}$ ($k = 1$).

In this case, $W(Y_{s,1}) = W(Y_{s-1,N}) \cup E^{s-1}$. Let $w \in W(p(V))$. There exists $x \in V$ such that $w \in p(x)$. If x is a terminal node, then there exists a play $\bar{w}$ such that $x = \{\bar{w}\}$ with $\bar{w} \in W^{s-1}$. If x is not terminal, choose any $\bar{w} \in x$. Note that w and $\bar{w}$ coincide up to bargaining round $s - 1$.

Since $W(V)$ is open, there are open sets of Ω, $U^t \times C^t$, $t = 1, \ldots, s$, such that $\bar{w} \in \left[\times_{t=1}^{s}\left(U^t \times C^t\right) \times (\Omega)^{\infty}\right] \cap \bar{W} \subseteq W(V)$. Consider the open basic set of Ω^{∞} given by $U = \left[\times_{t=1}^{s-1}\left(U^t \times C^t\right) \times (\Omega)^{\infty}\right]$. It follows that $U \cap \bar{W}$ is open in $\bar{W}$, hence in W, with $w \in U \cap \bar{W} \subseteq W(p(V))$, demonstrating that the latter set is open. □

This completes the verification of the hypotheses of the existence theorem in Alós-Ferrer and Ritzberger (2016b, Theorem 1; 2016a, Theorem 7.4).

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